Live Experiment

Periodicity Hubs with Discontinuous Spirals in a Noiseless Duffing Proxy: Experiment and Theory

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Abstract

We report the experimental and numerical observation of discontinuous spirals made of periodic oscillations in the control parameter space of an electronic circuit. Heretofore, only continuous spirals were known. Discontinuous spirals also whirl around an exceptional focal point, a hub which organizes very regularly the dynamics over wide portions of the parameter space. There is no mathematical theory yet to predict such hubs and their spirals.

Overview

We found a *discontinuous* spiral made of periodic oscillations in the control parameter space of the electronic circuit shown schematically on the right, which is a slight variation of an autonomous Duffing-like proxy introduced recently [1].

The circuit leads to an autonomous flow:

$$\dot{x}=y,\quad \dot{y}=x-x^{3}+by-kz,\quad \dot{z}=\omega(y-z) \qquad (1)$$

The existence of exceptional points organizing all oscillations into *continuous* spirals in the parameter space was reported a few years ago [2], shortly afterwards, such hubs were also observed in several other nonlinear systems such as semiconductor lasers, light-emitting diodes with optoelectronic feedback, chemical oscillators, Rössler oscillators, and other paradigmatic flows, e.g. [3].

Our circuit however shows regions of periodic oscillations in control parameter space which are also organized around a hub, but form a *discontinuous* spiral that involve an infinite alternation of periodicity islands with terminations looking like boomerangs, cusps and fishes. To pass from one periodicity island to the next one, it is always necessary to cross the surrounding chaotic phase.

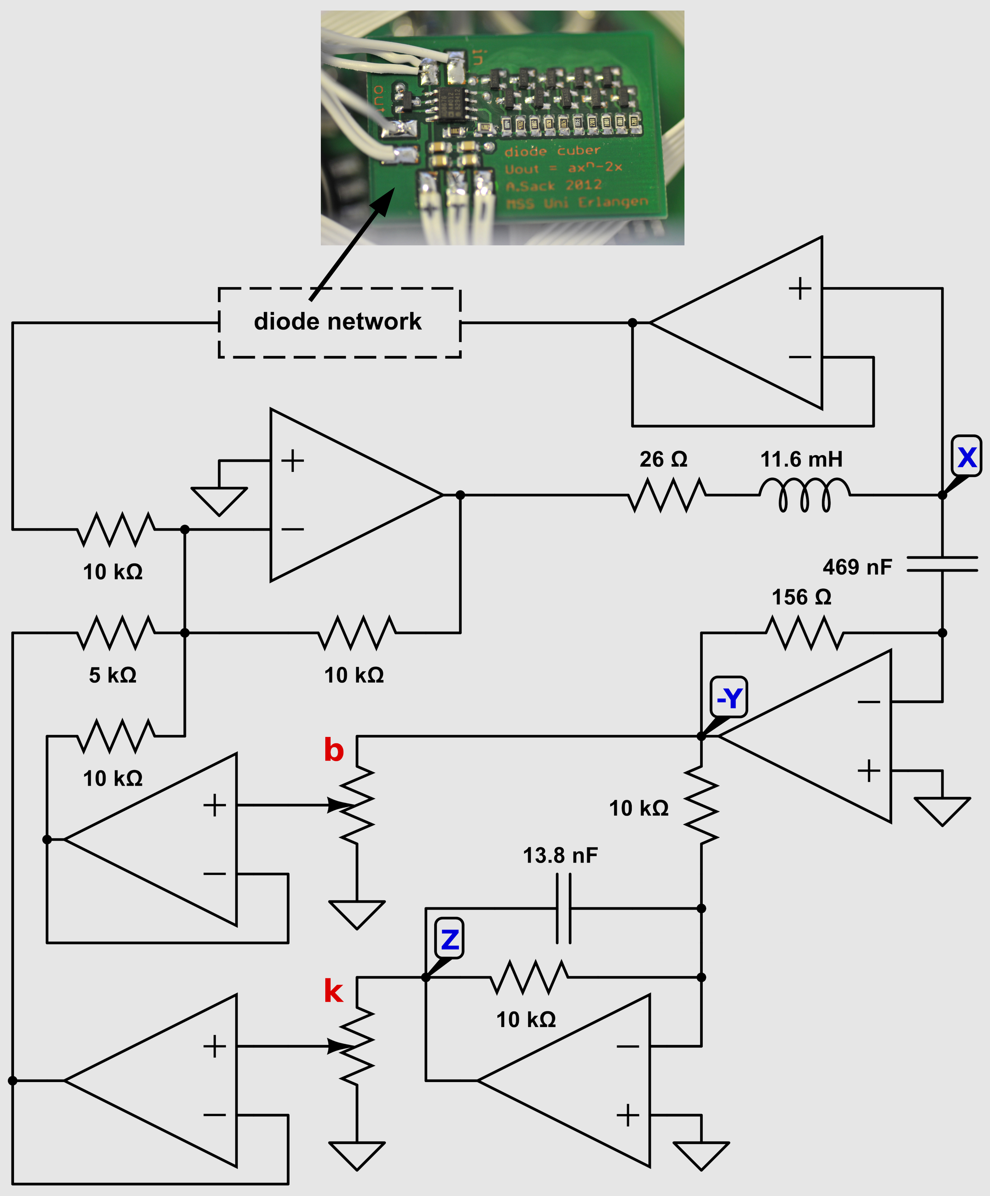
Circuit

The circuit on the right illustrates the implementation of Eqs. (1) and allows setting the values of **b** and **k** independently by means of digitally controlled attenuators. For all measurements we fixed ω = 0.5. A diode network is producing the required cubic function (small picture). The operational amplifiers are fast, precise and of low-noise, the resistors (except the ones in the diode network) are of low tolerance and low temperature coefficient. The voltages on the points **X** and **Z** in the figure were recorded using a PC with integrated analog-to-digital converter.

Procedure to record the behavior of the circuit for a given pair of parameters **b** and **k**:

- set **b** = 0 for about 10 ms to allow any oscillation to decay
- set **b**, **k** to the desired values
- wait for 0.2 s to allow transients to die out
- record the voltages **X**, **Z** for 0.5 s
- compute a 2D histogram of **X**, **Z** with a binning of 500 × 500
- count the number of bins containing non-zero values. They quantify the occupied fraction of phase-space.

This procedure was repeated for a mesh of equally spaced points in parameter space. Afterwards the values quantifying the occupied phase-space were normalized and used to build a heat-map, namely a graphical representation of the data where the individual values contained in a matrix of points are represented as colors.



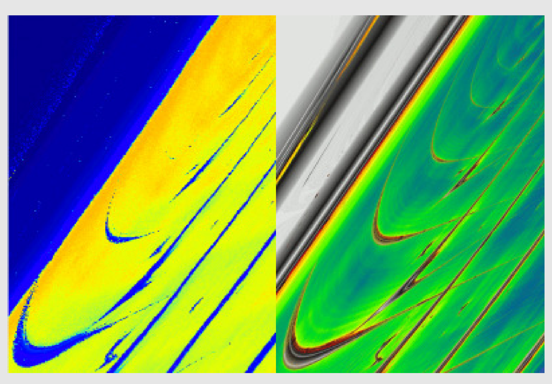
Simulation and Results

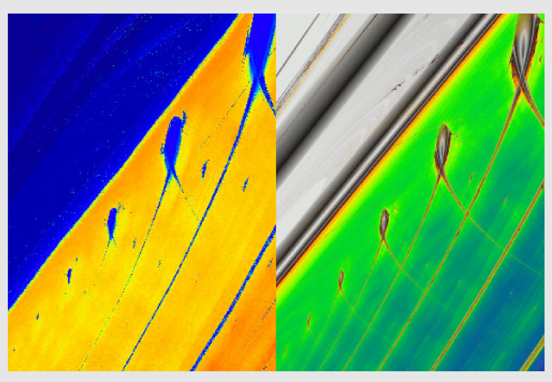
As an independent check of the experimental results, we computed a Lyapunov stability diagram for Eqs. (1) by solving them numerically using a standard fourth-order Runge-Kutta algorithm with fixed-step, h = 0.0045, over a mesh of equally spaced points. The first 70 × 10^3 time-steps were discarded as due to transient behavior. Along with the Jacobian of the system, the subsequent 1.4 × 10^6 time-steps were used to compute the Lyapunov spectrum of the oscillator. The computation of high-resolution stability diagrams is numerically a quite demanding task which we performed on 700 high-performance processors of an SGI Altix cluster with a theoretical peak performance of 16 Tflops.

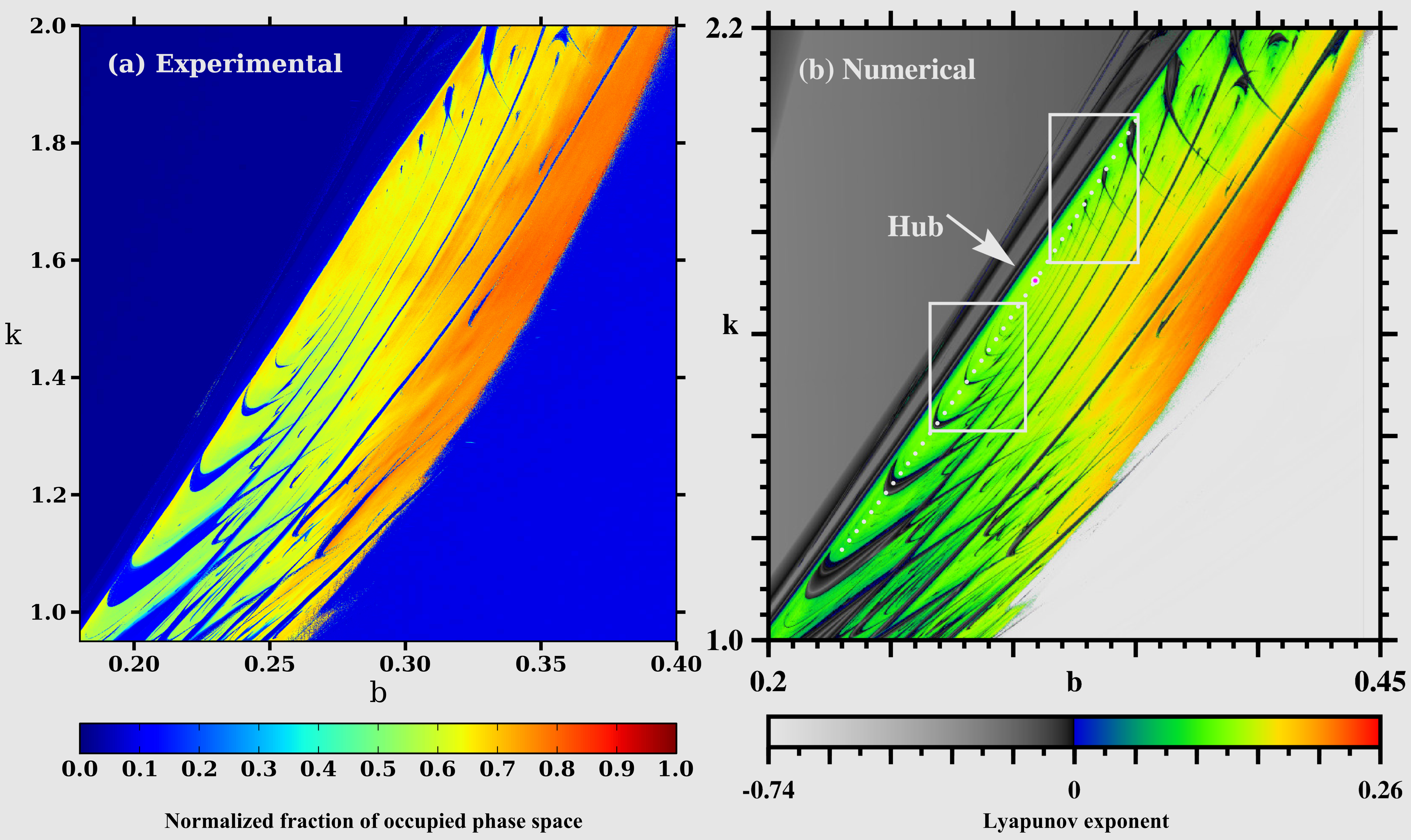
Figure (a) shows the normalized heat-map of the experiment with dark blue colors indicating a low complexity of the waveform and green/red colors implying a chaotic behavior.

Figure (b) shows a Lyapunov phase diagram illustrating how periodic and chaotic solutions self-organize over a wide region of the control parameter space. As indicated by the colorbar, colors represent chaotic phases (i.e. positive Lyapunov exponents) while darker shadings denote periodic oscillations (negative exponents). Both figures substantiate excellent agreement found between our experimental results and simulations.

The small figures below show crops of the experimental data and high resolution renderings of the numerical Lyapunov phase diagram as denoted by the white boxes in figure (b).







Summary

We observed experimentally and numerically a remarkable family of discontinuous spirals in the control parameter space of an electronic circuit. This shows that periodic oscillations may form either continuous or discontinuous spirals in stability diagrams. Discontinuous spirals also whirl around an exceptional focal point, a hub which organizes very regularly the dynamics over wide portions of the parameter space. We remark that there is no mathematical theory yet to predict such hubs and their spirals. We hope the discovery of discontinuous spirals to motivate the investigation of the mathematical conditions underlying their genesis.

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[2] C. Bonatto and J.A.C. Gallas, Phys. Rev. Lett. 101, 054101 (2008)
[3] R. Stoop, P. Benner, and Y. Uwate, Phys Rev. Lett. 105, 074102 (2010)

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